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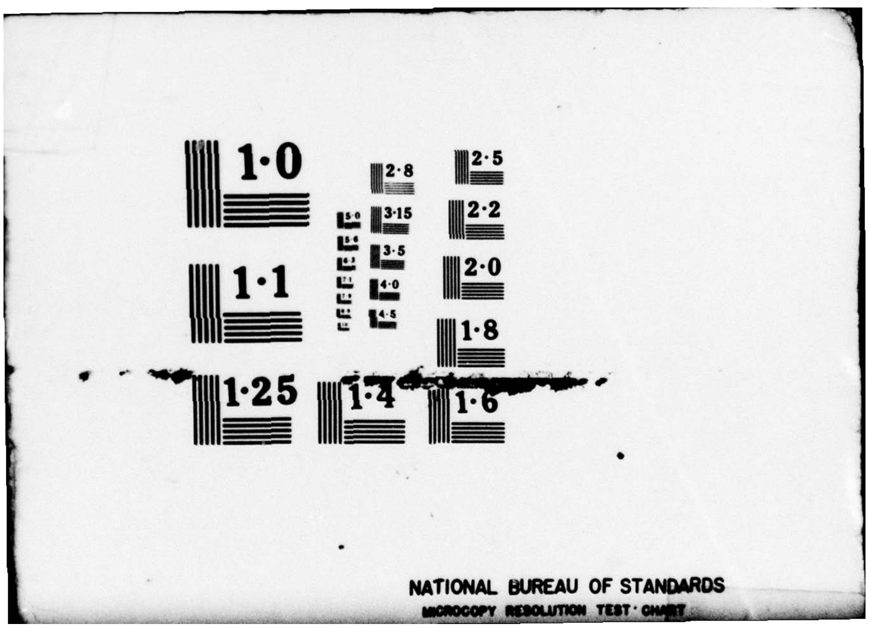
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MONOTONICITY IS NECESSARY AND SUFFICIENT FOR COMPENSATED COMPACTNESS

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ABSTRACT

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map and set $G_F \equiv \{v \in W^{1,\infty}(\mathbb{R}^n) \,|\, \operatorname{div}(F(\nabla v)) = 0\}$. The function F gives rise to a map $T_F : G_F \to (L^\infty(\mathbb{R}^n))^n$ via $T_F : v \to F(\nabla v)$.

We show that if T_F is continuous (for the w^* topologies on G_F and $(L^\infty(\mathbb{R}^n))^n$) then F is either affine or monotone.

AMS (MOS) Subject Classification: 47H05

Key Words: Monotonicity, w^* topology, Affine map, Affine subspace

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SIGNIFICANCE AND EXPLANATION

In the analysis of such nonlinear problems as the melting (Stefan) problem and the filtration of water through a porous dam, it is important to know whether small changes in the physics of the system produces large changes in the physical end-results. When the system is modelled mathematically this means that we have to study the behavior of some associated nonlinear operator. This is done by establishing a measure of the effect of small changes in the parameters in the problem (a topology). It turns out that in the two physical problems mentioned above, the nonlinear operators involved are monotone. Roughly speaking "$F(x)$ monotone increasing" means that an increase in x produces an increase in $F(x)$. In many situations, monotonicity of the operators involved guarantees that the corresponding physical problems are well-behaved, and guarantees convergence of numerical methods for solving these problems. Another important class of well-behaved operators are those arising from so-called affine mappings, i.e. linear operators plus a displacement.

The question naturally arises as to whether there are any other well-behaved classes of operators. In this paper the question is answered in the negative: if a given operator is well-behaved (e.g. in the sense that it has certain convergence properties) then it must be either monotone or affine.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

MONOTONICITY IS NECESSARY AND SUFFICIENT FOR COMPENSATED COMPACTNESS

Robert Jensen

0. Introduction

It has been known for a long time that monotone operators exhibit a kind of weak continuity. Namely, suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and monotone. If for a sequence $\{v_j\}_{j=1}^{\infty} \subset W^{1,\infty}(\mathbb{R}^n)$ we know

$$v_j \xrightarrow{w^*} v_0, \quad \text{div } F(\nabla v_j) = 0, \quad \nabla v_j = \left(\frac{\partial v_j}{\partial x_1}, \ldots, \frac{\partial v_j}{\partial x_n}\right),$$

and

$$F(\nabla v_j) \xrightarrow{w^*} u_0$$

then

$$F(\nabla v_0) = u_0 .$$

In this paper we show that the only nonlinear functions, F, which have the above property (with no additional restrictions on the sequences $\{v_j\}_{j=1}^{\infty}$ and $\{F(\nabla v_j)\}_{j=1}^{\infty}$) are exactly the monotone functions.

Section 1 is a recap of known results; Theorem 1.5 is a result brought to our attention by L. Tartar. This result reduces the analysis problem to a geometric one. Section 2 is the solution of the geometric problem.

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1. Some Analysis Results

Consider the following assumptions:

$$(1.1)\qquad \begin{cases} \{\bar{v}^j\}_{j=1}^{\infty} \subset (L^{\infty}(\mathbb{R}^n))^n, \quad \bar{v}^j \xrightarrow{w^*} \bar{v}^0 \text{ and} \\ \{\frac{\partial}{\partial x_k} v_\ell^j - \frac{\partial}{\partial x_\ell} v_k^j\} \text{ is uniformly bounded in } L^{\infty} \\ \text{for } 1 \le \ell,k \le n \text{ and } j \in \mathbb{Z}^+, \ (\bar{v}^j = (v_1^j,\dots,v_n^j)) , \end{cases}$$

$$(1.2)\qquad \begin{cases} F \in (C(\mathbb{R}^n))^n, \quad \bar{u}^j \equiv F(\bar{v}^j) \xrightarrow{w^*} \bar{u}^0 \text{ and} \\ \{\operatorname{div} \bar{u}^j\}_{j=1}^{\infty} \text{ is uniformly bounded in } L^{\infty}(\mathbb{R}^n) . \end{cases}$$

Let us set

$$(1.3)\qquad \begin{aligned} \mathcal{F} \equiv \{F : \mathbb{R}^n \to \mathbb{R}^n \,|\, &\text{If } F, \{\bar{v}^j\}_{j=0}^{\infty} \text{ and} \\ &\{\bar{u}^j\}_{j=0}^{\infty} \text{ satisfy (1.1) and (1.2) then } F(\bar{v}^0) = \bar{u}^0\} . \end{aligned}$$

The purpose of this paper is to give a useful characterization of $\mathcal{F}$.

Theorem 1.4. If F is affine or monotone then $F \in \mathcal{F}$.

Remark. The result is obvious for an affine map F. If F is monotone then Minty's argument can be used to prove the theorem. (For the sake of completeness we give a proof in the appendix.)

Theorem 1.5*. Let $F \in \mathcal{F}$. If

(1.6) $\langle F(b) - F(a), b - a\rangle = 0$ ($\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{R}^n$) ,

then

(1.7) $F(\alpha b + (1-\alpha)a) = \alpha F(b) + (1-\alpha)F(a)$ for $\alpha \in [0,1]$.

Proof. If $b = a$ then there is nothing to prove; so we now assume $b \neq a$.

For $\theta \in (0,1)$ we wish to construct a sequence of functions $\{w^j(x)\}_{j=1}^{\infty} \subset C^{0,1}(\mathbb{R}^n)$ such that

$$\nabla w^j(x) = \beta^j(\langle x, b-a\rangle)(b-a) + a$$

---

*We wish to thank L. Tartar for showing us this result.

where

$$\beta^j(s) \equiv \begin{cases} 0 & \text{if } \dfrac{s}{\|b-a\|} \in \left[\dfrac{k}{j}, \dfrac{k+\theta}{j}\right) \text{ for some } k \in \mathbb{Z} \\ 1 & \text{if } \dfrac{s}{\|b-a\|} \in \left[\dfrac{k+\theta}{j}, \dfrac{k+1}{j}\right) \text{ for some } k \in \mathbb{Z} \end{cases} .$$

To prove the existence of the functions w^j we shall assume j is fixed and suppress j in the following argument. We let T be any orthonormal transformation such that $T(b-a) = \|b-a\|e_1$ $(e_i = (\delta_{1i},\ldots,\delta_{ii},\ldots,\delta_{ni})$ for δ_{ji} the Kronecker-δ). We define a function $q(y)$ by

(1.8) $$q(y) = \int_0^{y_1} \beta(s)ds .$$

Therefore

$$\nabla q(y) = \beta(\langle y,e_1\rangle)e_1$$

so for

$$w'(x) = q(Tx)$$

we get

$$\nabla w'(x) = \nabla q(Tx)T = T^{-1}\nabla q(T(x)) = \beta(\langle x,T^{-1}(e_1)\rangle)T^{-1}(e_1)$$

$$= \beta\left(\langle x, \frac{b-a}{\|b-a\|}\rangle\right)\left(\frac{b-a}{\|b-a\|}\right) .$$

Setting $w^{\#}(x) = w'(\|b-a\|x)$ we arrive at

(1.9) $$\nabla w^{\#}(x) = \beta(\langle x,b-a\rangle)(b-a) .$$

Finally, we may define $w(x)$ by

(1.10) $$w(x) = w^{\#}(x) + \langle a,x\rangle .$$

This proves the existence of the desired sequence. It is easily seen that

(1.11) $$\nabla w^j(x) \xrightarrow{w^*} \theta a + (1-\theta)b ,$$

(1.12) $$F(\nabla w^j(x)) \xrightarrow{w^*} \theta F(a) + (1-\theta)F(b)$$

and with $\sigma^j \equiv \nabla w^j$

(1.13) $$\left\{\frac{\partial}{\partial x_k} v_\ell^j - \frac{\partial}{\partial x_\ell} v_k^j\right\} = 0 \text{ for all } 1 \le \ell, k \le n, \quad j \in \mathbb{Z}^+ .$$

Since $F \in \mathcal{F}$ it will follow from (1.11)-(1.13) that if

(1.14) $$\operatorname{div}(F(\bar{v}^j)) = 0 \text{ for all } j \in \mathbb{Z}^+$$

then $F(\theta a + (1-\theta)b) = \theta F(a) + (1-\theta)F(b)$.

In order to prove (1.14) we must show that for all $\phi \in C_0^\infty(\mathbb{R}^n)$

(1.15) $$\int_{\mathbb{R}^n} \langle F(\bar{v}^j), \nabla\phi\rangle(x)dx = 0 .$$

For convenience we again suppress the j and we consider for $\phi \in C_0^\infty(\mathbb{R}^n)$

$$E_\phi \equiv \int_{\mathbb{R}^n} \langle F(\bar{v}), \nabla\phi\rangle(x)dx .$$

We note that

$$E_\phi = \int_{\mathbb{R}^n} \langle F(\bar{v}) - F(a), \nabla\phi\rangle(x)dx ,$$

$$= \int_{\mathbb{R}^n} \langle \beta(\langle x, b-a\rangle)(F(b) - F(a)), \nabla\phi(x)\rangle dx .$$

Let T *be the same transformation as defined earlier in this proof.* Then for $Tx = y$ we get

$$E_\phi = \int_{\mathbb{R}^n} \langle T(\beta(\langle y, T(b-a)\rangle)(F(b) - F(a))), T(\nabla\phi(T^{-1}(y)))\rangle dy$$

$$= \int_{\mathbb{R}^n} \langle \beta(\langle y, \|b-a\|e_1\rangle)T(F(b) - F(a)), T(\nabla\phi(T^{-1}(y)))\rangle dy .$$

If $\psi(y) \equiv \phi(T^{-1}(y))$ then $\nabla\psi(y) = T\nabla\phi(T^{-1}(y))$ and so

$$E_\phi = \int_{\mathbb{R}^n} \beta(\langle y, \|b-a\|e_1\rangle)\langle T(F(b) - F(a)), \nabla\psi(y)\rangle dy$$

$$= \int_{\mathbb{R}^n} \beta(\langle y, \|b-a\|e_1\rangle)\langle T(F(b) - F(a)), \frac{\partial}{\partial y_1}\psi(y)e_1\rangle dy$$

since in all other directions the integrals are zero. Thus

$$E_\phi = \int_{\mathbb{R}^n} \beta(\langle y, \|b-a\|e_1\rangle) \frac{\partial}{\partial y_1}(\psi(y))\langle F(b) - F(a), b-a\rangle dy = 0 .$$

This proves $\operatorname{div}(F(\bar{v}^j)) = 0$ for all $j \in \mathbb{Z}^+$ and so completes the proof of the theorem.

2. Geometric Considerations

We make the following assumptions throughout this section:

(2.1) $$F : \mathbb{R}^n \to \mathbb{R}^n \quad \text{is continuous} .$$

(2.2) $$\left[\begin{array}{l} \text{If } a,b \in \mathbb{R}^n \text{ are vectors such that } \langle F(b) - F(a), b - a\rangle = 0 \\ \text{then } F(\alpha a + (1-\alpha)b) = \alpha F(a) + (1-\alpha)F(b) \text{ for } \alpha \in [0,1] . \end{array}\right.$$

Definition. For $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n \setminus \{0\}$ and $t \in \mathbb{R}$ we set

(2.3) $$G(x,y,t) \equiv \langle F(x + ty), y\rangle .$$

Lemma 2.4. If $G(x,y,t_1) = G(x,y,t_2)$ then

(2.5) $$F(x + ty) = \left(\frac{t_2 - t}{t_2 - t_1}\right) F(x + t_1 y) + \left(\frac{t - t_1}{t_2 - t_1}\right) F(x + t_2 y)$$

for $t \in [t_1, t_2]$.

Proof. By the definition (2.3) and the hypothesis of the lemma

$$\langle F(x + t_2 y) - F(x + t_1 y), (t_2 - t_1)y\rangle = 0 .$$

Therefore, for $\alpha \in [0,1]$ by (2.2)

$$F(x + \alpha t_1 y + (1 - \alpha)t_2 y) = \alpha F(x + t_1 y) + (1 - \alpha)F(x + t_2 y) .$$

With $\alpha = \dfrac{t_2 - t}{t_2 - t_1}$ the result claimed in the lemma follows easily.

Corollary 2.6. For fixed $x,y \in \mathbb{R}^n$, $G(x,y,t)$ is a monotone function in t.

Proof. If $G(x,y,t)$ is not monotone then there is a point t_0 where $G(x,y,t_0)$ is a strict local extremum. Thus, there are points $t_1 < t_0 < t_2$ such that

$$G(x,y,t_1) = G(x,y,t_2) .$$

By Lemma 2.4, definition (2.3) and the above we conclude that

$$G(x,y,t) = \text{constant for } t \in [t_1, t_2] .$$

This is a contradiction since $G(x,y,t_0)$ is then not a strict local extremum.

Therefore $G(x,y,t)$ must be monotone in t.

Lemma 2.7. If $x_1 + ty_1$ and $x_2 + sy_2$ describe the same line then $G(x_1,y_1,t)$ and $G(x_2,y_2,s)$ have the same monotonicity.

Proof. Since we get the same line from $x_1 + ty_1$ and from $x_2 + sy_2$ we conclude $y_1 = ky_2$ for some $k \neq 0$. Further we have $x_1 = x_2 + s_0y_2$ for some $s_0 \in \mathbb{R}$. Therefore $x_1 + ty_1 = x_2 + (s_0 + tk)y_2$ and by (2.3)

$$G(x_1,y_1,t) = kG(x_2,y_2,s_0 + tk) .$$

From here the conclusion of the lemma is clear.

By Lemma 2.7 we may unambiguously speak of the monotonicity of G on the line $x + ty$.

Lemma 2.8. Let $V \subset \mathbb{R}^n$ be a two-dimensional affine subspace, $V = z_0 + W$ where W is a linear subspace and consider G restricted to $V \times W \times \mathbb{R}$. Assume $x_1, x_2 \in V$ and $y_1, y_2 \in W$ are such that

$$G(x_1,y_1,t) \text{ is increasing and nonconstant}$$

and

$$G(x_2,y_2,t) \text{ is decreasing and nonconstant} .$$

Then F restricted to V is an affine map.

Proof. We may assume *w.ℓ.o.g.* that $x_1 = x_2$. Indeed by Lemma 2.7 if the lines $x_1 + ty_1$ and $x_2 + sy_2$ intersect we may parameterize in such a way that $x_1 = x_2$. Furthermore, since by hypothesis $G(x_1,y_1,t)$ is monotone increasing and nonconstant there is a nbhd. N_1 of y_1 such that $G(x_1,y,t)$ is monotone increasing and nonconstant for $y \in N_1$. Clearly for some $y \in N_1$ the lines $x_1 + ty$ and $x_2 + sy_2$ do intersect.

We let $x_0 = x_1$ and consider first the lines $x_0 + t(\alpha y_1 + (1 - \alpha)y_2)$. For some $\alpha_0 \in (0,1)$

$$G(x_0,\alpha_0y_1 + (1 - \alpha_0)y_2,t) = \text{constant (as a function of } t)$$

by the continuity of G and Corollary 2.6. Similarly, there is an $\alpha_1 \in (0,1)$ such that

$$G(x_0,-\alpha_1y_1 + (1 - \alpha_1)y_2,t) = \text{constant (as a function of } t) .$$

Let us denote $\alpha_0y_1 + (1 - \alpha_0)y_2$ and $-\alpha_1y_1 + (1 - \alpha_1)y_2$ by $\bar{y}_1$ and $\bar{y}_2$ respectively (note $\bar{y}_1$ and $\bar{y}_2$ are not parallel).

By Lemma 2.4 we have from the above that

$$F(x_0 + t\bar{y}_i) = (1 - \frac{t}{t_2})F(x_0) + \frac{t}{t_2} F(x + t_2\bar{y}_i) .$$

Holding $t = 1$ and letting $t_2 \to \infty$ we find

$$F(x_0 + \bar{y}_i) = F(x_0) + \lim_{t_2 \to \infty} \left(\frac{1}{t_2} F(x + t_2\bar{y}_i)\right) .$$

From this we conclude

$$F(x_0 + t\bar{y}_i) = F(x_0) + t(F(x_0 + \bar{y}_i) - F(x_0)) .$$

Now, consider any line in V which is determined by the two points $x_0 + \alpha_1\bar{y}_1$, $x_0 + \alpha_2\bar{y}_2$, $\alpha_i \neq 0$, $i = 1,2$ and so can be parameterized as $x_0 + \alpha_1\bar{y}_1 + t(\alpha_2\bar{y}_2 - \alpha_1\bar{y}_1)$. We now evaluate $G(x_0 + \alpha_1\bar{y}_1, \alpha_2\bar{y}_2 - \alpha_1\bar{y}_1, t)$ for $t \in \{0,1\}$. We find

$$G_0 \equiv G(x_0 + \alpha_1\bar{y}_1, \alpha_2\bar{y}_2 - \alpha_1\bar{y}_1, 0) = \langle F(x_0) + \alpha_1(F(x_0 + \bar{y}_1) - F(x_0)), \alpha_2\bar{y}_2 - \alpha_1\bar{y}_1\rangle$$

and

$$G_1 \equiv G(x_0 + \alpha_1\bar{y}_1, \alpha_2\bar{y}_2 - \alpha_1\bar{y}_1, 1) = \langle F(x_0) + \alpha_2(F(x_0 + \bar{y}_2) - F(x_0)), \alpha_2\bar{y}_2 - \alpha_1\bar{y}_1\rangle .$$

Thus

$$G_1 - G_0 = \langle \alpha_2(F(x_0 + \bar{y}_2) - F(x_0)) - \alpha_1(F(x_0 + \bar{y}_1) - F(x_0)), \alpha_2\bar{y}_2 - \alpha_1\bar{y}_1\rangle .$$

By expanding the right hand side of the above inequality we find

$$G_1 - G_0 = -\alpha_1\alpha_2(\langle F(x_0 + \bar{y}_2) - F(x_0), \bar{y}_1\rangle + \langle F(x_0 + \bar{y}_1) - F(x_0), \bar{y}_2\rangle) .$$

Since $\alpha_1, \alpha_2 \in \mathbb{R}\setminus\{0\}$ this implies that either

(2.9) $$G_1 - G_0 = 0 \text{ for all } \alpha_1 \text{ and } \alpha_2$$

or

(2.10) $$G_1 - G_0 \neq 0 \text{ for all } \alpha_1 \text{ and } \alpha_2 .$$

If (2.9) is the case then by Lemma 2.4 we can conclude that F restricted to V is an affine map.

If (2.10) holds we can conclude the following intermediate result:

(2.11) For any $x_1, x_2 \in V$, $y \in W$, $G(x_1,y,t)$ and $G(x_2,y,t)$ have the same monotonicity.

Indeed, if x_0 doesn't lie on either line this is simply a use of (2.10) and the fact

that if (α_1,α_2) is proportional to $(\bar{\alpha}_1,\bar{\alpha}_2)$ then $\operatorname{sgn}(\alpha_1\alpha_2) = \operatorname{sgn}(\bar{\alpha}_1\bar{\alpha}_2)$. In the remaining cases we simply use the continuity of G and a limiting argument.

Now, by (2.11) we conclude that for any $x \in \mathbb{R}^n$

$$G(x,\bar{y}_i,t) \equiv \text{constant (as a function of } t) .$$

Similar to a previous argument we also find that

$$F(x + t\bar{y}_i) = F(x) + t(F(x + \bar{y}_i) - F(x)) .$$

The proof of our lemma is almost complete now. Any point, z, in V can be written as

$$z = x_0 + t\bar{y}_1 + s\bar{y}_2 ,$$

and

$$\begin{aligned} F(x_0 + t\bar{y}_1 + s\bar{y}_2) &= F(x_0 + t\bar{y}_1) + s(F(x_0 + t\bar{y}_1 + \bar{y}_2) - F(x_0 + t\bar{y}_1) \\ &= F(x_0) + t(F(x_0 + \bar{y}_1) - F(x_0))) \\ &\quad + s(F(x_0 + \bar{y}_2) + t(F(x_0 + \bar{y}_1 + \bar{y}_2) - F(x_0 + \bar{y}_2)) \\ &\quad - (F(x_0) + t(F(x_0 + \bar{y}_1) - F(x_0)))) . \end{aligned}$$

We find

$$F(x_0 + t\bar{y}_1 + s\bar{y}_2) = A + sB + tC + tsD$$

where $A,B,C,D \in \mathbb{R}^n$.

Finally, it is not too difficult to see that since (2.2) holds for F then $D = 0$ and so F is an affine map.

This completes the proof of the lemma.

Lemma 2.12. Let V be defined as before and suppose that F restricted to V is a monotone function. Then for any $x_1,x_2 \in V$, $y \in W$, $G(x_1,y,t)$ and $G(x_2,y,t)$ have the same monotonicity (as functions of t). In particular, if one of the two functions is constant so is the other.

Proof. The only time this lemma is of any interest is in the case $G(x_1,y,t) = \text{constant}$ (as a function of t). Indeed, in this case we must show that $G(x_2,y,t) = \text{constant}$

(as a function of t). In order to do this we consider the functions

$$G(x_2, \frac{x_1}{t} + y - \frac{x_2}{t}, s) \quad \text{for} \quad t \in \mathbb{R} .$$

We use the fact that by Lemma 2.4 (as in the proof of Lemma 2.8)

$$F(x_1 + ty) = F(x_1) + t(F(x_1 + y) - F(x_1)) .$$

From this and the fact that $G(x,y,t) =$ constant in t we find

$$\langle F(x_1 + y) - F(x_1), y \rangle = 0 . \tag{2.13}$$

Next we evaluate $G(x_2, \frac{x_1}{t} + y - \frac{x_2}{t}, s)$. For $s = 0$,

$$G(x_2, \frac{x_1}{t} + y - \frac{x_2}{t}, 0) = \langle F(x_2), \frac{x_1}{t} + y - \frac{x_2}{t} \rangle ,$$

$$= \langle F(x_2), y \rangle + \frac{1}{t} \langle F(x_2), x_1 - x_2 \rangle ,$$

while for $s = t$ we get

$$G(x_2, \frac{x_1}{t} + y - \frac{x_2}{t}, t) = \langle F(x_1 + ty), \frac{x_1}{t} + y - \frac{x_2}{t} \rangle$$

$$= \langle F(x_1) + t(F(x_1 + y) - F(x_1)), \frac{x_1}{t} + y - \frac{x_2}{t} \rangle$$

$$= \langle F(x_1), y \rangle + \frac{1}{t} \langle F(x_1), x_1 - x_2 \rangle + \langle F(x_1 + y) - F(x_1), x_1 - x_2 \rangle$$

(the other terms add up to zero by 2.13). We conclude from the above that if $t \geq s > 0$, then

$$G(x_2, \frac{x_1}{t} + y - \frac{x_2}{t}, s) - G(x_2, \frac{x_1}{t} + y - \frac{x_2}{t}, 0) \leq$$

$$\langle F(x_1) - F(x_2), y \rangle + \frac{1}{t} \langle F(x_1) - F(x_2), x_1 - x_2 \rangle + \langle F(x_1 + y) - F(x_1), x_1 - x_2 \rangle$$

and if $0 > -s \geq -t$ then

$$G(x_2, -\frac{x_1}{t} + y + \frac{x_2}{t}, 0) - G(x_2, -\frac{x_1}{t} + y + \frac{x_2}{t}, -s) \leq$$

$$\langle F(x_2) - F(x_1), y \rangle + \frac{1}{t} \langle F(x_1) - F(x_2), x_1 - x_2 \rangle + \langle F(x_1) - F(x_1 + y), x_1 - x_2 \rangle .$$

Adding the two inequalities gives us for $0 < |s| \leq t$

$$G(x_2, \frac{x_1}{t} + y - \frac{x_2}{t}, s) - G(x_2, -\frac{x_1}{t} + y + \frac{x_2}{t}, -s) \leq \frac{2}{t}\langle F(x_1) - F(x_2), x_1 - x_2\rangle .$$

By letting $t \to \infty$ and using the continuity of G we get

$$G(x_2,y,s) - G(x_2,y,-s) \leq 0 .$$

Since by assumption F is monotone and therefore $G(x_2,y,t)$ is monotone increasing (as a function of t) this proves

$$G(x_2,y,t) = \text{constant in } t .$$

Corollary 2.14. Let V be defined as before and suppose that $-F$ restricted to V is a monotone function. Then for any $x_1,x_2 \in V$, $y \in W$, $G(x_1,y,t)$ and $G(x_2,y,t)$ have the same monotonicity (as functions of t).

Proof. Simply apply the previous lemma to $-F$. ($-F$ also satisfies (2.1) and (2.2).)

Lemma 2.15. For any $x_1,x_2,y \in \mathbb{R}^n$ the functions $G(x_1,y,t)$ and $G(x_2,y,t)$ have the same monotonicity as functions of t. In particular, if one is constant both are constant.

Proof. A unique two-dimensional affine subspace is determined by x_1, x_2 and y, namely

$$V \equiv \{x | x = x_1 + ty + s(x_2 - x_1) \text{ for } t,s \in \mathbb{R}\} .$$

Note that both of the lines $x_1 + ty$ and $x_2 + sy$ are in V.

If either F or $-F$ restricted to V is monotone then by either Lemma 2.12 or Corollary 2.14 (resp.) we see that Lemma 2.15 is valid.

If F is not monotone and $-F$ is not monotone then we can satisfy the hypothesis of Lemma 2.8. Thus F restricted to V is an affine map and again Lemma 2.15 is valid.

This completes the proof of the lemma.

Theorem 2.16. The function F is either monotone or $-F$ is monotone or F is an affine map.

Proof. By Lemma 2.8 the result is true for $n = 2$. We proceed by induction. Suppose Theorem 2.16 is valid for $n = m - 1$ and therefore the statement of Theorem 2.16 is valid on any $m - 1$ dimensional affine subspace $V \subset \mathbb{R}^m$.

We proceed with the proof of the theorem. If F or $-F$ is monotone on $\mathbb{R}^m$ we are done. Therefore we can find four vectors x_1, y_1, x_2 and y_2 such that

$$G(x_1, y_1, t) \text{ is increasing and nonconstant}$$

and

$$G(x_2, y_2, t) \text{ is decreasing and nonconstant .}$$

It is important now to note that by Lemma 2.15 we can assume $x_1 = x_2 = x_0$.

Let V be an $m - 1$ dimensional affine subspace containing the lines $x_0 + ty_1$ and $x_0 + ty_2$. By the induction hypothesis F restricted to V is an affine map. By using Lemma 2.15 again along with the induction hypothesis we see that F is an affine map when restricted to any of the affine subspaces $x + V$. Thus there are $m - 1$ linearly independent vectors $\{\tilde{y}_1, \ldots, \tilde{y}_{m-1}\}$ such that for any $x \in \mathbb{R}^m$

$$\text{(2.17)} \qquad F(x + \alpha_1\tilde{y}_1 + \cdots + \alpha_{m-1}\tilde{y}_{m-1}) = F(x) + \sum_{j=1}^{m-1} \alpha_j(F(x + \tilde{y}_j) - F(x)) .$$

Let y_m^* be any nonzero vector $y_m^* \notin \mathrm{Span}\{\tilde{y}_1, \ldots, \tilde{y}_{m-1}\}$. We consider $G(x_0, y_m^*, t)$; we claim we can find a nonzero vector $\tilde{y}_m$ such that $\tilde{y}_m \notin \mathrm{Span}\{\tilde{y}_1, \ldots, \tilde{y}_{m-1}\}$ and $G(x_0, \tilde{y}_m, t) = \text{constant}$ (as a function of t). Indeed, if $G(x_0, y_m^*, t) = \text{constant}$ we take $\tilde{y}_m = y_m^*$. If this is not the case we may assume $G(x_0, y_m^*, t)$ is increasing and nonconstant (as a function of t). Therefore, for some $\alpha_0 \in (0,1)$, $G(x_0, \alpha_0 y_m^* + (1 - \alpha_0)y_2, t) = \text{constant}$ (as a function of t). We may take $\tilde{y}_m = \alpha_0 y_m^* + (1 - \alpha_0)y_2$.

We know by Lemma 2.15 that for any $x \in \mathbb{R}^n$,

$$F(x + \alpha_m\tilde{y}_m) = F(x) + \alpha_m(F(x + \tilde{y}_m) - F(x)) .$$

Combining (2.17) with the above we see that

$$\begin{aligned}
F(\alpha_1\tilde{y}_1 + \cdots + \alpha_{m-1}\tilde{y}_{m-1} + \alpha_m\tilde{y}_m) &= F(\alpha_m\tilde{y}_m) + \sum_{j=1}^{m-1} \alpha_j(F(\alpha_m\tilde{y}_m + \tilde{y}_j) - F(\alpha\tilde{y}_m)) \\
&= F(0) + \alpha_m(F(\tilde{y}_m) - F(0)) + \sum_{j=1}^{m-1} \alpha_j(F(\tilde{y}_j) + \alpha_m(F(\tilde{y}_m + \tilde{y}_j) - F(\tilde{y}_j)) \\
&\qquad - F(0) - \alpha_m(F(\tilde{y}_m) - F(0)) \\
&= F(0) + \sum_{j=1}^{m} \alpha_j(F(\tilde{y}_j) - F(0)) + \sum_{j=1}^{m-1} \alpha_j\alpha_m(F(\tilde{y}_m + \tilde{y}_j) - F(\tilde{y}_j) - F(\tilde{y}_m) + F(0)) .
\end{aligned}$$

However, now it is an easy matter to see that

$$(F(\tilde{y}_m + \tilde{y}_j) - F(\tilde{y}_m) - F(\tilde{y}_j) + F(0)) = 0 \quad \text{for all} \quad j = 1,\ldots,m-1 .$$

Thus

$$F(\alpha_1\tilde{y}_1 + \cdots + \alpha_m\tilde{y}_m) = F(0) + \sum_{j=1}^{m} \alpha_j(F(\tilde{y}_j) - F(0)) .$$

This completes the proof of the theorem by induction.

We are now able to state the main result of this paper.

Theorem 2.18. Let $\mathcal{F}$ be defined as in section 1 then

$$\mathcal{F} = \mathcal{M} \equiv \{F : \mathbb{R}^n \to \mathbb{R}^n \,|\, F \text{ is an affine map or either } F \text{ or } -F \text{ is a continuous monotone map}\} .$$

Proof. By Theorem 1.4 $\mathcal{F} \supset \mathcal{M}$. By Theorem 1.7 if $F \in \mathcal{F}$ then F satisfies (2.1) and (2.2). Therefore by Theorem 2.16

$$\mathcal{F} \subset \mathcal{M} .$$

This completes the proof of the theorem.

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Appendix

We give here a proof of Theorem 1.4. As already remarked we need only consider the case: F is continuous and monotone.

Proof of Theorem 1.4. We shall use a lemma.

Lemma A.1. Let $\bar{v} \in (L^\infty(\mathbb{R}^n))^n$ and such that

$$\{\frac{\partial}{\partial x_k} \phi - \frac{\partial}{\partial x_\ell} v_k\} = f_{k\ell} \in L^\infty(\mathbb{R}^n) \quad \text{for all} \quad 1 \le k,\ell \le n$$

and where $\bar{v} = (v_1,\ldots,v_n)$. Then

$$\bar{v} = \nabla\phi + \bar{w}$$

where $\phi \in C^{0,1}(\mathbb{R}^n)$, $\phi(0) = 0$ and

$$|\phi(x) - \phi(y)| \le K_1 \|\bar{v}\|_{(L^\infty(\mathbb{R}^n))^n} |x - y|$$

and where $\bar{w} \in (C^{0,1}(\mathbb{R}^n))^n$, $\bar{w}(0) = \bar{0}$ and

$$\|w_r(x) - w_r(y)\| \le K_2(\max\{\|f_{k\ell}\|_{L^\infty(\mathbb{R}^n)}\})|x - y|$$

for $\bar{w} = (w_1,\ldots,w_n)$ and $1 \le r \le n$.

Proof. We set

$$\phi(x_1,\ldots,x_n) = \sum_{r=1}^{n} \int_0^{x_r} v_r(x_1,\ldots,x_{r-1},s,0,\ldots,0)ds .$$

Clearly ϕ has the required properties; so next we consider $\frac{\partial}{\partial x_r}\phi - v_r$. Using our hypothesis on $\bar{v}$ we see

$$\frac{\partial}{\partial x_r}\phi(x) - v_r(x) = \sum_{k=r+1}^{n} \int_0^{x_k} f_{kr}(x_1,\ldots,x_{k-1},s,0,\ldots,0)ds .$$

From the equation above it is easily seen that $\bar{w} = \bar{v} - \nabla\phi$ also has the required properties. This completes the proof of the lemma.

We continue now with the proof of the theorem. For $\bar{v}^j,\bar{u}^j$ as given in Theorem 1.4 we see that if $\phi \in C_0^\infty(\mathbb{R}^n)$ then

(A.2) $$(F(\bar{v}^j) - \bar{u}^j, \phi(\bar{\psi} - \bar{v}^j))_I \ge 0 \quad \text{for all} \quad \bar{\psi} \in (C^\infty(\mathbb{R}^n))^n ,$$

where $\langle \bar{f},\bar{g}\rangle_I = \int_{\mathbb{R}^n} \langle \bar{f}(x),\bar{g}(x)\rangle dx$ for $\bar{f}$ and $\bar{g}$ for which the integral above is defined.

By the monotonicity of F and (A.2) we get

$$\langle F(\bar{\psi}) - \bar{u}^j, \phi(\bar{\psi} - \bar{v}^j)\rangle \geq 0 \quad \text{for all } \bar{\psi} \in (C^\infty(\mathbb{R}^n))^n .$$

Upon letting $j \to \infty$ we get

(A.3) $\langle F(\bar{\psi}),\phi(\bar{\psi} - \bar{v}^0)\rangle - \langle \bar{u}^0,\phi\bar{\psi}\rangle + \liminf_{j\to\infty} \langle \bar{u}^j,\phi\bar{v}^j\rangle \geq 0$ for all $\bar{\psi} \in (C^\infty(\mathbb{R}^n))^n$.

Let $\bar{v}^j = \nabla\tilde{\phi}^j + \bar{w}^j$ with ϕ^j and $\bar{w}^j$ as given by Lemma A.1. Then

$$\langle \bar{u}^j,\phi\bar{v}^j\rangle = \langle \bar{u}^j,\phi(\nabla\tilde{\phi}^j + \bar{w}^j)\rangle = -\langle \operatorname{div} \bar{u}^j,\phi\tilde{\phi}^j\rangle - \langle \bar{u}^j,(\nabla\phi)\tilde{\phi}^j\rangle + \langle \bar{u}^j,\phi\bar{w}^j\rangle .$$

Since $\phi\tilde{\phi}^j \to \phi\tilde{\phi}^0$ in $L^1(\mathbb{R}^n)\phi\tilde{w}^j \to \phi\tilde{w}^0$ uniformly and $(\nabla\phi)\tilde{\phi}^j \to (\nabla\phi)\tilde{\phi}^0$ also in $L^1(\mathbb{R}^n)$ we see that

$$\begin{aligned}\liminf_{j\to\infty} \langle \bar{u}^j,\phi\bar{v}^j\rangle &= -\langle \operatorname{div} \bar{u}^0,\phi\tilde{\phi}^0\rangle - \langle \bar{u}^j,(\nabla\phi)\tilde{\phi}^0\rangle + \langle \bar{u}^j,\phi\bar{w}\rangle \\ &= \langle \bar{u}^0,\phi(\nabla\tilde{\phi}^0 + \bar{w}^0)\rangle = \langle \bar{u}^0,\phi\bar{v}^0\rangle .\end{aligned}$$

Combining this with (A.3) yields

(A.4) $$\langle F(\bar{\psi}) - \bar{u}^0,\phi(\bar{\psi} - \bar{v}^0)\rangle \geq 0 \quad \text{for all } \bar{\psi} \in (C^\infty(\mathbb{R}^n))^n .$$

Let $\bar{\sigma}^k \to \bar{v}^0$ in $L^1_{loc}(\mathbb{R}^n)$ such that

$$\|\phi(\bar{\sigma}^k - \bar{v}^0)\|_{(L^\infty(\mathbb{R}^n))^n} \leq \frac{1}{k^2} .$$

Choose $\bar{\psi} = \frac{1}{k}\bar{\psi}' + (1 - \frac{1}{k})\bar{\sigma}^k$ and using this in (A.4) gives

$$\langle F(\tfrac{1}{k}\bar{\psi}' + (1 - \tfrac{1}{k})\bar{\sigma}^k) - \bar{u}^0, \tfrac{1}{k}\phi(\bar{\psi}' - \bar{v}^0)\rangle + (1 - \tfrac{1}{k})\langle F(\tfrac{1}{k}\bar{\psi}' + (1 - \tfrac{1}{k})\bar{\sigma}^k),\phi(\bar{\sigma}^k - \bar{v}^0)\rangle \geq 0 .$$

Multiplying the inequality above by k gives us

$$\langle F(\tfrac{1}{k}\bar{\psi}' + (1 - \tfrac{1}{k})\bar{\sigma}^k) - \bar{u}^0,\phi(\bar{\psi}' - \bar{v}^0)\rangle + (1 - \tfrac{1}{k})K\tfrac{1}{k} \geq 0$$

for some K independent of k. By letting $k \to \infty$ we get

$$\langle F(\bar{v}^0) - \bar{u}^0,\phi(\bar{\psi}' - \bar{v}^0)\rangle \geq 0 \quad \text{for all } \bar{\psi}' \in (C^\infty(\mathbb{R}^n))^n .$$

Since $\phi \in C_0^\infty$ is arbitrary also this implies

$$F(\bar{v}^0) = \bar{u}^0 .$$

This completes the proof of Theorem 1.4.

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Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map and set $G_F \equiv \{v \in W^{1,\infty}(\mathbb{R}^n) \mid \operatorname{div}(F(\nabla v)) = 0\}$. The function F gives rise to a map $T_F : G_F \to (L^\infty(\mathbb{R}^n))^n$ via $T_F : v \to F(\nabla v)$.

We show that if T_F is continuous (for the w^* topologies on G_F and $(L^\infty(\mathbb{R}^n))^n$) then F is either affine or monotone.

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